Journal of Global Optimization **18:** 357–365, 2000.

A Simple Finite Cone Covering Algorithm for Concave Minimization

CHRISTOPHE MEYER
Institut für Mathematik B, TU Graz, Steyrergasse 30, A-8010 Graz, Austria (e-mail: meyer@opt.math.tu-graz.ac.at)

Abstract. In 1964, in a seminal paper, Tuy proposed a simple algorithm for concave minimization over a polytope. This algorithm was shown to cycle some years later. Recently however it has been shown that despite this possibility of cycling, Tuy's algorithm always finds the optimal solution of the problem. We present a modification of it which simplifies the cycle detection.

Key words: Concave minimization, Cone covering algorithm

1. Introduction

The concave minimization problem

$$(CP) \qquad \min\{f(x) \mid x \in P\}$$

consists in finding a global minimizer x^* of the concave function f over the polytope $P = \{x \in \mathbb{R}^n \ : \ Ax \leqslant b\}$ where A is a matrix of $\mathbb{R}^{m \times n}$ and b a vector of $\mathbb{R}^m$.

The core problem for concave minimization is the problem of *transcending the incumbent* (see Horst and Tuy [3], Tuy [11]). Given a number γ, it can be defined as follows:

core(γ, P**)** : check whether $f(x) \geqslant \gamma$ for all $x \in P$, and if not, find a point $x' \in P$ such that $f(x') < \gamma$.

An optimal solution for problem (CP) can be found by solving a finite number of times the core problem as follows (see, e.g., Tuy [11]). Start with an extreme point $\overline{x}$ of P, and let $\overline{f}$ be its value. Solve problem $core(\overline{f}, P)$. If the result is that $f(x) \geqslant \overline{f}$ for all $x \in P$, $\overline{x}$ is an optimal solution of problem (CP). If on the other hand, a point $x' \in P$ is found such that $f(x') < \overline{f}$, we look for a new extreme point $\overline{x}'$ of P such that $f(\overline{x}') \leqslant f(x')$. This can be done for example by using the Caratheodory theorem (see [8]), or if the gradient of the function at a point can be easily computed, by solving a linear program (see, e.g., Tuy [11, p. 135]). This procedure generates a strictly decreasing sequence $\{\overline{f}^k\}$, each value of it corresponding to a different extreme point of P. Since P has a finite number of extreme points, this procedure must be finite.

In [10], Tuy proposed several ideas and algorithms for solving this core problem, that still influence today's methods (for surveys on algorithms for concave minimization, see e.g., Horst and Tuy [4], Benson [2]). One of these algorithms, which consists in covering the polytope by (possibly overlapping) cones, was shown to cycle by Zwart [12]. To avoid cycling, a small modification [13, 1] was brought to Tuy's algorithm, which results in a cone partitioning algorithm (i.e., with no two cones intersecting, except possibly at their boundaries). In the next 20 years, a lot of works has been done on cone partitioning algorithms, mainly developing special subdivision procedures to ensure convergence. Recently, Locatelli [6] and Jaumard and Meyer [5] showed independently and by different approaches, that these procedures are not necessary to obtain the convergence. On the other hand, rather surprisingly, it was shown in [7] that the original algorithm of Tuy, despite it may cycle, always finds an optimal solution of problem (CP). In [8], a non-cycling cone covering algorithm was proposed. This algorithm uses a slightly different kind of cones, with possibly more than n extreme rays. At each iteration, up to $n + 1$ linear programs have to be solved.

In this paper we propose a simple cone covering algorithm that generates cones with non-increasing value. This algorithm uses traditional cones with n extreme rays and requires only 2 linear programs per cone. Cycling may be possible only between cones of same value, which facilitates its detection. The paper is organized as follows. In Section 2, we describe the basic operations. The algorithm is given in Section 3 and its finiteness and correctness are proved in Section 4. A short discussion concludes the paper in Section 5.

2. Basic operations

2.1. INITIAL COVERING

Let O be an extreme point of the polytope. For simplicity, we assume that O is non-degenerated (if no non-degenerated extreme point is available, we can start the algorithm with an initial cone partition formed by $n + 1$ cones: see, e.g., [7]). The algorithm starts with the cone K^0 of origin O and whose n extreme rays are determined by the n adjacent extreme points to O. Since O is a feasible point, we can assume without loss of generality that $\gamma \leqslant f(O)$. Indeed, if this is not the case, O solves the core problem.

2.2. ELIMINATION TEST

The elimination test is the standard one of the literature (see, e.g., Horst and Tuy [4]). Let $K = cone\{O; x^1, \ldots, x^n\}$ be a cone, where x^j, $j = 1, \ldots, n$ are n extreme points of P. For each j, we compute the so-called *γ-extension* $y^j = \theta_j x^j$, which is the intersection of the halfline $[Ox^j)$ with the boundary of the level set $C_\gamma = \{x \in \mathbb{R}^n : f(x) \geqslant \gamma\}$ (if C_γ is not bounded, we consider instead $C_\gamma \cap B$

where B is a ball of center O containing P). Consider the following linear program:

$$LP1(K) \quad \max \quad \sum_{j=1}^{n} \frac{\lambda_j}{\theta_j}$$
$$\text{s.t.} \quad \begin{cases} \sum_{j=1}^{n} \lambda_j A x^j \leqslant b \\ \lambda \geqslant 0 \end{cases}$$

and its dual:

$$DLP1(K) \quad \min \quad \mu^t b$$
$$\text{s.t.} \quad \begin{cases} \mu^t A x^j \geqslant \frac{1}{\theta_j}, \quad j = 1, \ldots, n \\ \mu \geqslant 0. \end{cases}$$

If the optimal value ρ^1 is less than or equal to 1, this means that the portion of the polytope P contained in the cone K is included in the simplex $S = conv\{O, y^1, y^2, \ldots, y^n\}$ as it can be seen by the substitutions $y^j = \frac{x^j}{\theta_j}$ in the constraints of $LP1(K)$. We then have

$$\min_{x \in K \cap P} f(x) \geqslant \min_{x \in S} f(x) = \min\{f(O), f(y^1), \ldots, f(y^n)\} \geqslant \gamma$$

which shows that $K \cap P$ cannot contain a point with value $< \gamma$. Therefore if $\rho^1 \leqslant 1$, we eliminate the cone.

2.3. SUBDIVISION

If $\rho^1 > 1$, we subdivide the cone K. Let λ^1 and μ^1 be respectively an optimal solution of the primal and of the dual. It was shown (see e.g. [5]) that $H = \{x \in \mathbb{R}^n : \mu^1 A x = \rho^1\}$ is a supporting hyperplane for the polytope, hence defines a face F of P. Note that since $\mu^1 \geqslant 0$ and $\rho^1 = \mu^1 b$, a definition of F is

$$F = \{x \in \mathbb{R}^n : a_i x = b_i \ \forall i \mid \mu_i^1 > 0; \ a_i x \leqslant b_i \ \forall i \mid \mu_i^1 = 0\}.$$

We take as subdivision point any extreme point $w^2 = \sum_{j=1}^{n} \lambda_j^2 x^j$ of F such that

$$\sum_{j=1}^{n} \frac{\lambda_j^2}{\theta_j} > 1. \tag{1}$$

Such a point can for example be found by solving the following linear program by a simplex-like algorithm:

$$LP2(K,\mu^1) \qquad \max \quad \sum_{j=1}^{n} \frac{\lambda_j}{\theta_j}$$
$$\text{s.t.} \quad \begin{cases} a_i x = b_i, & \forall i \mid \mu_i^1 > 0 \\ a_i x \leqslant b_i, & \forall i \mid \mu_i^1 = 0 \\ x = \sum_{j=1}^{n} \lambda_j x^j \end{cases}$$

(this linear program is feasible since it admits the solution $(x = \sum_{j=1}^{n} \lambda_j^1 x^j, \lambda = \lambda^1)$ and its optimal value is greater than 1; moreover if (x, λ) is an extreme point solution, then x is an extreme point of F by definition of the constraints).

Note that the point w^2 is not necessarily in K since we do not require $\lambda^2 \geqslant 0$, and is an extreme point of P since it is an extreme point of one of its faces. It will be tested for a possible solution to the core problem: indeed if $f(w^2) < \gamma$, the core problem is solved with $x' = w^2$.

The subdivision process is defined as follows. For each j such that $\lambda_j^2 > 0$, we construct the cone K^j, obtained from K by replacing x^j by w^2. It is easy to show that the union of all these cones K^j forms a cover of the cone K (see, e.g., [7, Proposition 1]), therefore we replace K by the cones K^j. We say that the cones K^j are the *sons* of cone K by the subdivision process.

The following result only uses the fact that w^2 is an extreme point of F.

PROPOSITION 1. *Let K' be a son of K by subdivision with respect to w^2. If $f(w^2) \geqslant \gamma$ and $\rho^1(K) > 1$, then $\rho^1(K') \leqslant \rho^1(K)$.*

Note that the assumptions are not restrictive since if $f(w^2) < \gamma$ the core problem is solved and if $\rho^1(K) \leqslant 1$ the cone K is deleted.

Proof. Since $f(w^2) \geqslant \gamma$, the γ-extension of w^2 is $y' = \theta w^2$ with $\theta \geqslant 1$. Let K' be a son of K: it differs from K by the replacement of one of the edge x^j, $j = 1, \ldots, n$ by w^2. We claim that μ^1 is a feasible solution of problem $DLP1(K')$ with value $\rho^1(K)$. Indeed, $\mu^1 A x^j \geqslant \frac{1}{\theta_j}$ for $j = 1, \ldots, n$ and $\mu^1 \geqslant 0$ as μ^1 is an optimal solution of $DLP1(K)$. Moreover $\mu^1 A w^2 = \mu^1 b = \rho^1(K) > 1 \geqslant \frac{1}{\theta}$. Since $DLP1(K')$ is a minimization problem, it follows that $\rho^1(K') \leqslant \rho^1(K)$.

3. Algorithm

We now give formally the algorithm for solving the core problem. The algorithm uses 3 sets: $\mathcal{C}over$ is the set of cones that have yet to be processed; $\mathcal{D}one$ is the set of cones that have been processed and whose value is equal to the current value $\overline{\rho}$: this set must be checked when generating a new cone in order to avoid regenerating an already processed cone. Finally, $\mathcal{N}ew$ is the set of new cones resulting from the subdivision of a cone.

Step 1 (initialization): construct an initial conical cover $\mathcal{N}ew$ of P as explained in Section 2.1. Compute the value of f at the origin and at the intersection points of the generating rays with the polytope: if one of these values is less than γ, stop and return the corresponding point. Initialize $\mathcal{C}over$ to $\emptyset$, $\mathcal{D}one$ to $\emptyset$ and $\overline{\rho}$ to $+\infty$.

Step 2 (fathoming): for each cone K in $\mathcal{N}ew$, compute the γ-extensions and solve the problem $DLP1(K)$, obtaining the optimal solution $\mu^1(K)$ and value $\rho^1(K)$. Remove from $\mathcal{N}ew$ all cones K for which $\rho^1(K) \leqslant 1$.

Step 3 (anti-cycling): $\mathcal{N}ew \leftarrow \mathcal{N}ew \setminus \mathcal{D}one$.

Step 4 (subdivision point computation): for each cone K in $\mathcal{N}ew$, choose an extreme point $w^2(K)$ of $F = P \cap \{x \in \mathbb{R}^n : \mu^1(K)Ax = \rho^1(K)\}$ satisfying (1). If $f(w^2(K)) < \gamma$: stop and return $x' = w^2(K)$.
Let $\mathcal{C}over \leftarrow \mathcal{C}over \cup \mathcal{N}ew$.

Step 5 (cone selection): if $\mathcal{C}over = \emptyset$, stop: $f(x) \geqslant \gamma$ for all $x \in P$. Otherwise select the cone $\tilde{K} = \arg\max_{K \in \mathcal{C}over} \rho^1(K)$ and remove it from $\mathcal{C}over$. If $\rho^1(\tilde{K}) < \overline{\rho}$, reset $\mathcal{D}one$ to $\emptyset$ and $\overline{\rho}$ to $\rho^1(\tilde{K})$. Add $\tilde{K}$ to $\mathcal{D}one$.

Step 6 (subdivision): subdivide the cone $\tilde{K}$ via the point $w^2(\tilde{K})$ as indicated in Section 2.3. Let $\mathcal{N}ew$ be the set of sons of $\tilde{K}$. Return to Step 2.

Note that in Step 3, it is only necessary to check if a cone of $\mathcal{N}ew$ is in $\mathcal{D}one$ if its value ρ is equal to the current value $\overline{\rho}$.

4. Convergence proof

A straightforward adaptation of the proof used to prove the validity of Tuy's algorithm (see [7]) shows that this modified algorithm solves the core problem.

THEOREM 1. *After a finite number of iterations, the above algorithm stops at Step 1, 4 or 5. In the first two cases, a point $x' \in P$ such that $f(x') < \gamma$ is found; in the last case, $f(x) \geqslant \gamma$ for all $x \in P$.*

Proof. Since the cones are defined by extreme points of P, which are in finite number, the number of possible cones is finite. Moreover the value ρ^1 associated to a cone depends only on that cone (note that γ is fixed). By Proposition 1 and the cone selection rule in Step 5, a cone with value ρ^1 greater than the current value $\overline{\rho}$ cannot be obtained. Moreover the anti-cycling rule in Step 3 ensures that a cone with value $\rho^1 = \overline{\rho}$ cannot be regenerated. Since at each iteration, a finite number of cones is generated, it follows that the algorithm will stop after a finite number of iterations.

If the algorithm stops at Step 1 or 4, clearly the returned point x' is feasible and satisfies $f(x') < \gamma$, so the core problem is solved. It remains to show that if there exists a point $x' \in P$ such that $f(x') < \gamma$, then the algorithm stops at Step 1 or 4.

Assume by contradiction that such a point x' exists but that the algorithm does not stop at Step 1 or 4. Let $\gamma' = f(x')$. Since f is concave on $\mathbb{R}^n$, it is also continuous (see, e.g., Rockafellar [9]), hence by definition

$$\forall \varepsilon > 0 \quad \exists\, \delta_\varepsilon > 0 \quad | \quad x \in B(x', \delta_\varepsilon) \;\Rightarrow\; |f(x) - \gamma'| < \varepsilon$$

where $B(x', \delta_\varepsilon)$ denote the ball of center x' and radius δ_ε.
We fix ε to $\frac{\gamma-\gamma'}{2}$, which implies that $f(x) \leqslant \frac{\gamma+\gamma'}{2} < \gamma$ for all $x \in B(x', \delta_\varepsilon)$. To any subset S_i of $vert(P)$ with $|S_i| < n$, we associate the affine subspace H_i spanned by O and by the elements of S_i. Since the H_i are in finite number and of dimension $< n$, the set $B(x', \delta_\varepsilon) \cap P \setminus \left(\bigcup_i H_i\right)$ is non-empty. Let $\tilde{x} \in B(x', \delta_\varepsilon) \cap P \setminus \left(\bigcup_i H_i\right)$: then $f(\tilde{x}) < \gamma$. We consider the sequence $\{K^q\}, q = 0, 1, \ldots$ of cones containing $\tilde{x}$ such that for all q, K^{q+1} is a son of K^q by the subdivision process of Step 6. Note that $\rho^1(K^q) > 1$ for all q, since otherwise $f(x) \geqslant \gamma$ for all $x \in K^q \cap P$, contradicting the fact that $f(\tilde{x}) < \gamma$. Hence all cones of the sequence belong to the set of cones generated by the algorithm. Moreover since the subdivision process is always defined, the sequence is infinite. We will now prove that all cones in this sequence must be distinct, which is not possible since the number of possible cones is finite: this contradiction will then imply that our assumption was false, i.e., that the algorithm must stop at Step 1 or 4. To show that all cones in the sequence are distinct, assume that $K^q = cone\{x^{q1}, \ldots, x^{qn}\}$ contains $\tilde{x}$, and let $y^{qj} = \theta^q_j x^{qj}$, $j = 1, \ldots, n$ be the γ-extensions. Since K^q is nondegenerate, there exists a unique $\tilde{\nu}^q > 0$ such that

$$\tilde{x} = \sum_{j=1}^{n} \tilde{\nu}^q_j x^{qj} \tag{2}$$

(the fact that no component of the vector is null is a consequence of the assumption $\tilde{x} \notin \bigcup_i H_i$). We claim that $\sum_{j=1}^{n} \frac{\tilde{\nu}^q_j}{\theta^q_j} > 1$. Indeed, if $\sum_{j=1}^{n} \frac{\tilde{\nu}^q_j}{\theta^q_j} \leqslant 1$, $\tilde{x} \in S^q = conv\{O, y^{q1}, \ldots, y^{qn}\}$, which by concavity of f and definition of the points y^{qj}, $j = 1, \ldots, n$, implies $f(\tilde{x}) \geqslant \gamma$, a contradiction. It is now not difficult to show that the cone K^{q+1} is the son of K^q obtained by replacing the point $x^{q\ell_q}$ by $w^q = w^2(K^q) = \sum_{j=1}^{n} \lambda^q_j x^{qj}$ with ℓ_q satisfying

$$\frac{\tilde{\nu}^q_{\ell_q}}{\lambda^q_{\ell_q}} = \min_{j|\lambda^q_j > 0} \left\{ \frac{\tilde{\nu}^q_j}{\lambda^q_j} \right\}.$$

We then have $K^{q+1} = cone\{x^{q+1,1}, \ldots, x^{q+1,n}\}$ with $x^{q+1,j} = x^{qj}$ for $j = 1, \ldots, n,\ j \neq \ell_q$ and $x^{q+1,\ell_q} = w^q$, and

$$\tilde{x} = \sum_{j=1}^{n} \tilde{\nu}_j^{q+1} x^{q+1,j}$$

with

$$\tilde{\nu}_j^{q+1} = \begin{cases} \tilde{\nu}_j^q - \dfrac{\tilde{\nu}_{\ell_q}^q}{\lambda_{\ell_q}^q}\lambda_j^q & \text{if } j \neq \ell_q \\ \dfrac{\tilde{\nu}_{\ell_q}^q}{\lambda_{\ell_q}^q} & \text{if } j = \ell_q. \end{cases}$$

If $f(w^q) < \gamma$, the algorithm would stop at Step 4, so we have $f(w^q) \geqslant \gamma$ for all q. Then $\theta_j^{q+1} = \theta_j^q$ for $j \neq \ell_q$ and $\theta_{\ell_q}^{q+1} \geqslant 1$, hence

$$\begin{aligned} \sum_{j=1}^{n} \frac{\tilde{\nu}_j^{q+1}}{\theta_j^{q+1}} = \sum_{j=1, j\neq\ell_q}^{n} \frac{\tilde{\nu}_j^{q+1}}{\theta_j^q} + \frac{\tilde{\nu}_{\ell_q}^{q+1}}{\theta_{\ell_q}^{q+1}} &= \sum_{j=1}^{n} \frac{1}{\theta_j^q}\left(\tilde{\nu}_j^q - \frac{\tilde{\nu}_{\ell_q}^q}{\lambda_{\ell_q}^q}\lambda_j^q\right) + \frac{\tilde{\nu}_{\ell_q}^q}{\theta_{\ell_q}^{q+1}\lambda_{\ell_q}^q} \\ &= \sum_{j=1}^{n} \frac{\tilde{\nu}_j^q}{\theta_j^q} - \frac{\tilde{\nu}_{\ell_q}^q}{\lambda_{\ell_q}^q}\left(\sum_{j=1}^{n} \frac{\lambda_j^q}{\theta_j^q} - \frac{1}{\theta_{\ell_q}^{q+1}}\right) \\ &< \sum_{j=1}^{n} \frac{\tilde{\nu}_j^q}{\theta_j^q}, \end{aligned}$$

the last inequality holding because λ^q satisfies (1), $\theta_{\ell_q}^{q+1} \geqslant 1$, and $\frac{\tilde{\nu}_{\ell_q}^q}{\lambda_{\ell_q}^q} > 0$ by the choice of $\tilde{x}$. Since the value of $\sum_{j=1}^{n} \frac{\tilde{\nu}_j^q}{\theta_j^q}$ depends only on $\tilde{x}$ and on the cone, this shows that a same cone cannot repeat, contradicting the finiteness of the number of distinct cones. Hence the algorithm must stop at Step 1 or 4.

5. Discussions

If the subdivision point is taken to be w^1 rather than w^2, we obtain the cone partitioning algorithm with a pure strategy of ω-subdivision [1, 13, 5, 6].

If we replace $LP2(K, \mu^1)$ by

$$LP2'(K) \quad \max \quad \sum_{j=1}^{n} \frac{\lambda_j}{\theta_j}$$

$$\text{s.t.} \quad \begin{cases} a_i x \leqslant b_i, & \forall i = 1, \ldots, m \\ x = \sum_{j=1}^{n} \lambda_j x^j. \end{cases}$$

(i.e., if no constraint of P is transformed to an equality), we obtain essentially Tuy's 1964 algorithm (actually in Tuy's algorithm, the elimination test is based on the optimal value of $LP2$ rather than $LP1$, which results in less eliminations; $LP1$ is not solved).

Tuy's 1964 algorithm and the cone partitioning algorithm only need to solve one linear problem per cone: in Tuy's algorithm it is $LP2'$ while in the cone partitioning one it is $LP1$. We observe that Tuy's algorithm is finite (or can easily be transformed to a finite one) but that it may waste a lot of time by reexploring large portion of the polytope (note in particular that there is no monotonicity result comparable to Proposition 1 for ρ^2, so a son can be worse than its father: see [7]). On the opposite, the cone partitioning algorithm never reexplores regions of the polytope that were already explored, but it is still not known if it is finite when an exact optimal solution is sought.

By considering a different linear program for each operation (deletion and determination of the subdivision point), we have obtained an intermediate algorithm: its finiteness is easy to ensure since we have only to detect possible cycles between cones of same value; and although some regions of the polytope may be reexplored, the monotonicity of ρ^1 limits this reexploration.

Currently, these 2 linear programs are defined using the same cone. In the future, we could possibly obtain an even better algorithm by solving $LP1$ on a tighter cone (not necessarily defined by extreme points), that does not include the non-necessary parts of the polytope added by the covering.

It is not known if cycling can really occur in our algorithm. However if we modify $LP2$ as indicated above (i.e., if we replace $LP2$ by $LP2'$), it is known that cycling can occur: see [7].

Finally, observe that the algorithm does not need to solve $LP2$ at optimality. A basic feasible solution with value strictly greater than 1 is sufficient.

Acknowledgments

This work was initiated while the author was at GERAD/École Polytechnique de Montréal, where it was supported by NSERC-network grant NET0200815. The final version of this paper has been prepared during the author's stay at the TU Graz, with financial support by the Spezialforschungsbereich F 003 "Optimierung und Kontrolle", Projektbereich Diskrete Optimierung.

References

1. Bali, S. (1973), 'Minimization of a Concave Function on a Bounded Convex Polyhedron'. Ph.D. thesis, University of California at Los Angeles.
2. Benson, H.P. (1996), 'Concave Minimization: Theory, Applications and Algorithms'. In: R. Horst and P.M. Pardalos (eds.), *Handbook of Global Optimization*. Kluwer Academic Publishers.

3. Horst, R. and Tuy, H. (1991), 'The Geometric Complementarity Problem and Transcending Stationarity in Global Optimization'. *DIMACS Series in Discrete Mathematics and Theoretical Computer Science* 4: 341–354.
4. Horst, R. and Tuy, H. (1996), *Global Optimization (Deterministic Approaches)*. Third, revised and enlarged edition. Springer-Verlag, Berlin.
5. Jaumard, B. and Meyer, C. (1998), 'A Simplified Convergence Proof for the Cone Partitioning Algorithm'. *Journal of Global Optimization* 13: 407–416.
6. Locatelli, M. (1999), 'Finiteness of Conical Algorithms with ω-subdivisions'. *Mathematical Programming* 85(3): 593–616.
7. Meyer, C. (1997), 'On Tuy's 1964 Cone Splitting Algorithm for Concave Minimization'. Les Cahiers du GERAD G-97-48, GERAD. http://www.crt.umontreal.ca/GERAD/cahiers/G9748.ps. To appear in *From Local to Global Optimization*, proceeding of the workshop in honor of Professor Tuy's 70th birthday (Linköping), Kluwer Academic Publishers.
8. Meyer, C. and Jaumard, B. (1998), 'A New Finite Cone Covering Algorithm for Concave Minimization'. Les Cahiers du GERAD G-98-70, GERAD.
9. Rockafellar, R.T. (1970), Convex Analysis. Princeton University Press, Princeton, New Jersey.
10. Tuy, H. (1964), 'Concave Programming under Linear Constraints'. *Soviet Mathematics* 5: 1437–1440.
11. Tuy, H. (1998), *Convex Analysis and Global Optimization*, Vol. 22 of *Nonconvex Optimization and its Applications*. Kluwer Academic Publishers.
12. Zwart, P.B. (1973), 'Nonlinear Programming: Counterexamples to Two Global Optimization Algorithms'. *Operations Research* 21: 1260–1266.
13. Zwart, P.B. (1974), 'Global Maximization of a Convex Function with Linear Inequality Constraints'. *Operations Research* 22: 602–609.